

A NOTE ON LINEAR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

SUNG KYU CHOI* AND NAMJIP KOO**

ABSTRACT. This paper deals with linear impulsive fractional differential equations involving the Caputo derivative with non-integer order q . We provide exact solutions of linear impulsive fractional differential equations with constant coefficient by mean of the Mittag-Leffler functions. Then we apply the exact solutions to improve impulsive integral inequalities with singularity.

1. Introduction

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals). In particular, this discipline involves the notion and methods of solving of fractional differential equations, i.e., differential equations involving fractional derivatives of the unknown function. Fractional differential equations are a generalization of differential equations through the application of fractional calculus. Recently, fractional differential equations play a significant role in modeling the anomalous dynamics of various processes related to complex systems in most areas of science and engineering.

The exponential function e^z plays a fundamental role in mathematics and it is really useful in theory of integer order differential equations. We can write it in a form of series:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)}.$$

Received August 22, 2015; Accepted September 09, 2015.

2010 Mathematics Subject Classification: Primary 26A33, 34A08, 34A37.

Key words and phrases: impulsive fractional differential equation, fractional integral inequality, Caputo fractional derivative, Mittag-Leffler function.

Correspondence should be addressed to Namjip Koo, [njoo@cmu.ac.kr](mailto:njkoo@cmu.ac.kr).

This work was supported by research fund of Chungnam National University.

The Mittag-Leffler functions which is the generalizations of exponential function play an important role in the theory of fractional differential equations.

We recall the notions of Mittag-Leffler functions which was originally introduced by G. M. Mittag-Leffler in 1902(see [9]). That is, the Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad \alpha > 0, z \in \mathbb{C}, \quad (1.1)$$

where Γ is the Gamma function given by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \operatorname{Re}(z) > 0.$$

Choi et al. [1] obtained an exact solution of linear Caputo fractional differential equation by the help of the Mittag-Leffler functions. Also, Choi et al. [2, 3] studied impulsive integral inequalities with a non-separable kernel and stability of Caputo fractional differential equations. Denton and Vatsala [4] established the explicit representation of the solution of the linear fractional differential equation with variable coefficient and they developed the Gronwall integral inequality for the Riemann-Liouville fractional differential equation.

Fečkan et al. [5] studied a Cauchy problem for a fractional differential equation with linear impulsive conditions and make a counterexample to illustrate the concepts of piecewise continuous solutions used in current papers are not appropriate. Also, Wang et al. [11] obtained many new existence, uniqueness and data dependence results of solutions for non-linear impulsive fractional differential equations with Caputo fractional derivative via some generalized singular Gronwall inequalities.

In this paper we provide an exact solution for a linear impulsive fractional differential equation with Caputo fractional derivative by mean of the Mittag-Leffler functions. Then we apply the exact solution to improve an impulsive integral inequality with singularity.

2. Main results

In this section we deal with linear impulsive Caputo fractional differential equations with constant coefficient. We present exact solutions of linear impulsive fractional differential equations with Caputo fractional derivative by the help of the Mittag-Leffler function. Also, we apply the exact solutions to obtain singular integral inequalities of Gronwall

type. For the general theory and applications of impulsive differential equations, we refer the reader to [7].

Let q be a positive real number such that $0 < q \leq 1$ and $t_0, T \in \mathbb{R}_+ = [0, \infty)$. We consider the following fractional Cauchy problems

$$\begin{cases} {}^C D_{t_0}^q u = f(t, u(t)), t \neq t_k, t \in J := [t_0, T], \\ \Delta u(t_k) := u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), k = 1, 2, \dots, m, \\ u(t_0) = u_0, \end{cases} \quad (2.1)$$

where ${}^C D_{t_0}^q$ is the Caputo fractional derivative of order q with the lower limit zero, $u_0 \in \mathbb{R}$, $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is jointly continuous, $I_k : \mathbb{R} \rightarrow \mathbb{R}$ and t_k satisfy $0 \leq t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $u(t_k^+) = \lim_{\varepsilon \rightarrow 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\varepsilon \rightarrow 0^-} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$. Denote by $C(J, \mathbb{R})$ the set of all continuous functions from J into \mathbb{R} . Also, let $PC(J, \mathbb{R})$ be the set of all functions from J into \mathbb{R} as follows:

$$PC(J, \mathbb{R}) = \{u : J \rightarrow \mathbb{R} \mid u \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, \dots, m, \text{ and there exist } u(t_k^-) \text{ and } u(t_k^+), k = 1, \dots, m, \text{ with } u(t_k^-) = u(t_k)\}.$$

For the fractional calculus and the theory of fractional differential equations, we refer the reader to [6, 8, 10].

We recall the definition of Caputo fractional derivative of a function $g : [t_0, \infty) \rightarrow \mathbb{R}$.

DEFINITION 2.1. [6] The Caputo fractional derivative of order q of a function g is defined by

$${}^C D_{t_0}^q g(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^t (t-s)^{-q} g'(s) ds,$$

where $g'(t) = \frac{dg(t)}{dt}$.

For the notion of solution and the existence of solutions for Equation (2.1), see [5, 11].

LEMMA 2.2. [11] A function $u \in C(J, \mathbb{R})$ is a solution of the fractional integral equation

$$u(t) = u_0 - \frac{1}{\Gamma(q)} \int_{t_0}^a (a-s)^{q-1} f(s, u(s)) ds + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s)) ds,$$

if and only if u is a solution of the following fractional Cauchy problems

$$\begin{cases} {}^C D_{t_0}^q u = f(t, u(t)), t \in J, \\ u(a) = u_0, a > t_0. \end{cases} \quad (2.2)$$

LEMMA 2.3. [11] *A function $u \in C(J, \mathbb{R})$ is a solution of the fractional integral equation*

$$u(t) = \begin{cases} u(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s)) ds, & t \in [t_0, t_1], \\ u(t_0) + \sum_{t_0 < t_k < t} I_k(u(t_k^-)) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s)) ds, & t \in (t_k, t_{k+1}], k = 1, \dots, m, \end{cases}$$

if and only if u is a solution of Equation (2.1).

We can obtain the following result on an exact solution of homogeneous linear impulsive fractional differential equations by the help of the Mittag-Leffler functions.

THEOREM 2.4. *If we set $f(t, u) = \lambda u$ and $I_k(u(t_k^-)) = \beta_k u(t_k^-)$, $k = 1, 2, \dots, m$, with constants λ and β_k in Equation (2.1), then the solution $u(t)$ of Equation (2.1) is given by*

$$u(t) = \begin{cases} u_0 E_q(\lambda(t-t_0)^q), & t \in [t_0, t_1], \\ u_0 \prod_{i=1}^k (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) E_q(\lambda(t-t_0)^q), & t \in (t_k, t_{k+1}], \\ & k = 1, 2, \dots, m. \end{cases}$$

Proof. Let $t \in [t_0, t_1]$. Then we have

$$u(t) = u(t_0) E_q(\lambda(t-t_0)^q), \quad t \in [t_0, t_1].$$

Let $t \in (t_1, t_2]$. By Lemma 2.2, we obtain

$$\begin{aligned} u(t) &= u(t_1^+) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1-s)^{q-1} \lambda u(s) ds + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} \lambda u(s) ds \\ &= (1 + \beta_1) u(t_1^-) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1-s)^{q-1} \lambda u(s) ds + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} \lambda u(s) ds \\ &= u(t_0) + \beta_1 u(t_1^-) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} \lambda u(s) ds \\ &= u(t_0) (1 + \beta_1 E_q(\lambda(t_1 - t_0)^q)) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} \lambda u(s) ds, \quad t \in (t_1, t_2]. \end{aligned}$$

Thus we have

$$u(t) = u(t_0) (1 + \beta_1 E_q(\lambda(t_1 - t_0)^q)) E_q(\lambda(t-t_0)^q), \quad t \in (t_1, t_2].$$

Let $t \in (t_2, t_3]$. From Lemma 2.2, we obtain

$$\begin{aligned} u(t) &= u(t_2^+) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_2 - s)^{q-1} \lambda u(s) ds + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} \lambda u(s) ds \\ &= (1 + \beta_2)u(t_2^-) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_2 - s)^{q-1} \lambda u(s) ds + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} \lambda u(s) ds \\ &= u(t_0) + \beta_1 u(t_1^-) + \beta_2 u(t_2^-) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} \lambda u(s) ds \\ &= u(t_0)[1 + \beta_1 E_q(\lambda(t_1 - t_0)^q) + \beta_2(1 + \beta_1 E_q(\lambda(t_1 - t_0)^q)) E_q(\lambda(t_2 - t_0)^q)] \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} \lambda u(s) ds \\ &= u(t_0) \prod_{i=1}^2 (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) E_q(\lambda(t - t_0)^q), \quad t \in (t_2, t_3]. \end{aligned}$$

Let $t \in (t_k, t_{k+1}]$. By above similar argument, we have

$$u(t) = u(t_0) \prod_{i=1}^k (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) E_q(\lambda(t - t_0)^q), \quad t \in (t_k, t_{k+1}],$$

where $k = 1, 2, \dots, m$. This completes the proof. □

To prove our main result, we need the following lemma on Caputo fractional integral inequality of Gronwall type which can be found in [1, 8, 12].

LEMMA 2.5. *Let $m \in C(J, \mathbb{R}_+)$ and suppose that*

$$m(t) \leq m_0 + \frac{\lambda}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} m(s) ds, \quad t_0 \leq t \leq T, \quad (2.3)$$

then $m(t) \leq m_0 E_q(\lambda(t - t_0)^q)$, $t_0 \leq t \leq T$, where m_0 and λ are nonnegative constants.

We can obtain the following impulsive fractional integral inequality by using Caputo fractional integral inequality of Gronwall type.

THEOREM 2.6. *Let $u \in PC(J, \mathbb{R}_+)$ satisfying the following inequality*

$$u(t) \leq c + \frac{\lambda}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} u(s) ds + \sum_{t_0 < t_k < t} \beta_k u(t_k^-), \quad t \geq t_0, \quad (2.4)$$

where c, λ and $\beta_k (k = 1, 2, \dots, m)$ are nonnegative constants. Then

$$u(t) \leq c \prod_{i=1}^k (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) E_q(\lambda(t - t_0)^q), t \in (t_k, t_{k+1}], \quad (2.5)$$

where $k = 1, 2, \dots, m$.

Proof. Let $t \in [t_0, T]$. It follows from Lemma 2.5 that

$$u(t) \leq c E_q(\lambda(t - t_0)^q), t \in [t_0, t_1], \quad (2.6)$$

$$u(t) \leq (c + \sum_{i=1}^k \beta_i u(t_i^-)) E_q(\lambda(t - t_0)^q), t \in (t_k, t_{k+1}], \quad (2.7)$$

where $k = 1, 2, \dots, m$.

Let $t \in (t_1, t_2]$. In view of (2.6), we have

$$\begin{aligned} u(t) &\leq (c + \beta_1 u(t_1^-)) E_q(\lambda(t - t_0)^q) \\ &= (c + \beta_1 c E_q(\lambda(t_1 - t_0)^q)) E_q(\lambda(t - t_0)^q) \\ &= c [1 + \beta_1 E_q(\lambda(t_1 - t_0)^q)] E_q(\lambda(t - t_0)^q), t \in (t_1, t_2]. \end{aligned} \quad (2.8)$$

Let $t \in (t_2, t_3]$. In view of (2.6) and (2.8), we have

$$\begin{aligned} u(t) &\leq [c + \beta_1 u(t_1^-) + \beta_2 u(t_2^-)] E_q(\lambda(t - t_0)^q) \\ &= [c + \beta_1 (c E_q(\lambda(t_1 - t_0)^q)) \\ &\quad + \beta_2 c (1 + \beta_1 E_q(\lambda(t_1 - t_0)^q)) E_q(\lambda(t_2 - t_0)^q)] E_q(\lambda(t - t_0)^q) \\ &= c [(1 + \beta_1 E_q(\lambda(t_1 - t_0)^q)) (1 + \beta_2 E_q(\lambda(t_2 - t_0)^q))] E_q(\lambda(t - t_0)^q) \\ &= c \left[\prod_{i=1}^2 (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) \right] E_q(\lambda(t - t_0)^q), t \in (t_2, t_3]. \end{aligned}$$

From above similar argument, we have

$$u(t) \leq c \left[\prod_{i=1}^k (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) \right] E_q(\lambda(t - t_0)^q), t \in (t_k, t_{k+1}],$$

where $k = 1, 2, \dots, m$. This completes the proof. □

We can obtain the following result in [11, Lemma 2.8] as a corollary of Lemma 2.7.

COROLLARY 2.7. *Let $u \in PC(J, \mathbb{R}_+)$ satisfy the following inequality*

$$u(t) \leq c_1(t) + \frac{\lambda}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} u(s) ds + \sum_{t_0 < t_k < t} \beta_k u(t_k^-), t \geq t_0,$$

where $c_1(t)$ is positive continuous and nondecreasing on J , and λ, β_k are nonnegative constants. Then

$$u(t) \leq c_1(t) \prod_{i=1}^k (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) E_q(\lambda(t - t_0)^q), \quad t \in (t_k, t_{k+1}],$$

where $k = 1, 2, \dots, m$.

Proof. Since $c_1(t)$ is positive and nondecreasing on J , we have

$$\frac{u(t)}{c_1(t)} \leq 1 + \frac{\lambda}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} \frac{u(s)}{c_1(s)} ds + \sum_{t_0 < t_k < t} \beta_k \frac{u(t_k^-)}{c_1(t_k^-)}, \quad t \in (t_k, t_{k+1}],$$

where $k = 1, 2, \dots, m$. From Theorem 2.6, we have

$$u(t) \leq c_1(t) \left[\prod_{i=1}^k (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) \right] E_q(\lambda(t - t_0)^q), \quad t \in (t_k, t_{k+1}],$$

where $k = 1, 2, \dots, m$. This completes the proof. \square

References

- [1] S. K. Choi, B. Kang, and N. Koo, Stability for Caputo fractional differential equations, *Proc. Jangjeon Math. Soc.* **16** (2013), 165-174.
- [2] S. K. Choi, B. Kang, and N. Koo, Stability of Caputo fractional differential systems, *Abstr. Appl. Anal.* **2014** (2014), Article ID 631419, 6 pages.
- [3] S. K. Choi, N. Koo, and C. Ryu, Impulsive integral inequalities with a non-separable kernel, *J. Chungcheong Math. Soc.* **27** (2014), 651-659.
- [4] Z. Denton and A. S. Vatsala, Fractional integral inequalities and applications, *Comput. Math. Appl.* **59** (2010), 1087-1094.
- [5] M. Fečkan, Y. Zhou, and J. Wang, On the concept and existence of solution for impulsive fractional differential equations, *Commun Nonlinear Sci Numer Simulat.* **17** (2012), 3050-3060.
- [6] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
- [7] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific Publishing Co. Pte. Ltd., NJ, 1989.
- [8] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers Ltd, 2009.
- [9] G. M. Mittag-Leffler, Sur l'intégrable de Laplace-Abel, *C. R. Acad. Sci. Paris (Ser. II)* **136** (1903), 937-939.
- [10] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [11] J. Wang, Y. Zhou, and M. Fečkan, Nonlinear impulsive problems for fractional differential equations and Ulam stability, *Comput. Math. Appl.* **64** (2012), 3389-3405.

- [12] H. Ye, J. Gao, and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equations *J. Math. Anal. Appl.* **328** (2007), 1075-1081.

*

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: sgchoi@cnu.ac.kr

**

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: njkoo@cnu.ac.kr